Symmetries of specific minimal gauged supergravities in 5 dimensions

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Outline of the talk

- Introduction in the study of internal symmetries of manifolds
- Definitions and properties of Killing-Yano and Stäckel-Killing tensors and Killing spinors
- Internal symmetries of minimal gauged supergravities in 5 dimensions and the Killing-Maxwell system
- Internal symmetries of the Chong-Cvetič-Lü-Pope black hole as an example of minimal gauged supergravity in 5 dimensions
Outline of the talk-continued

- Killing spinors of the black hole solution Houri-Takeuchi-Yasui as an example of minimal gauged supergravity in 5 dimensions with a Sasaki structure deformed by torsion
- The equivalence of conformal Killing-Yano tensors and Killing spinors for Sasaki-Einstein manifolds
- Conclusions

Based on:

C. Rugina, "Comments on the symmetries of minimal gauged supergravities in 5 dimensions with Killing-Yano torsion” (submitted to Class. Quant. Grav.)
Introduction to the study of internal symmetries of manifolds
About manifolds and internal symmetries

- Hidden symmetries (internal) are characterized by Killing-Yano and Stäckel-Killing tensors.
- Spacetime symmetries (isometries) are characterized by Killing vectors.
- Both types of symmetries are important for solving the dynamics of the system.
Motivation for the study of the Killing-Yano and Stäckel-Killing tensors

• These tensors are correlated with internal symmetries and they facilitate the integration of the equations of motion.

• They are useful in separating variables and solving the Dirac and Klein-Gordon equations in curved spacetimes (with torsion also).

• These tensors are naturally associated with supersymmetries and they connect classical and quantum symmetries.

• These tensors lead to a better understanding of the physics of black holes of various types and the associated symmetries in an arbitrary number of dimensions.

• They lead to the construction of Dirac-type operators, which are conserved and generate new superalgebras.
Motivations to study Killing-Yano and Stäckel-Killing tensors

- These tensors indicate the presence of supersymmetries in a semi-classical system (such as particle with spin and charged particle with spin).

- These tensors help with the construction of the Killing spinors, very well known for classifying the vacuums of various supergravities.

- They lead to the discovery and understanding of new hidden symmetries of the physical systems.
Definitions and properties of Killing-Yano and Stäckel-Killing tensors and Killing spinors
Definition of Killing-Yano tensors

Definition

A differential form of order $p$, $Y \in \Omega^p(M)$ is a Killing-Yano tensor if $\nabla Y \in \Omega^{p+1}(M)$, i.e. $\nabla_\mu Y_{\alpha_1\alpha_2\ldots\alpha_p}$ is totally antisymmetric.

Remarks and conventions:

- $M$ is a Riemannian manifold and $\nabla$ is the associated Levi-Civita connection

- the Killing-Yano tensors depend on the metric of the manifold and we shall assume that the torsion of the connection is null and that $\nabla g = 0$
Definition of Stäckel-Killing tensors

**Definition**

A contravariant totally symmetric tensor $K$ of rang $p$ on a manifold $\mathcal{M}$ is a Stäckel-Killing tensor if

$$\nabla (\mu K^\alpha_1^\alpha_2^\cdots^\alpha_p) = 0$$

**Properties of Killing vectors - which are tensors of rank 1:**

- Killing vectors are in one-to-one correspondence with continuous symmetries of the respective manifold.

- The existence of each Killing vector assumes the existence of some conserved quantities, associated with a geodesic motion, such that the metric doesn’t change in the direction of the Killing vector.
Conformal Stäckel-Killing tensors and conformal Killing-Yano tensors

**Definition**

A totally symmetric tensor is a conformal Stäckel-Killing tensor if it obeys the following equation:

\[
K(\alpha_1\alpha_2\cdots\alpha_p;\beta) = g_\beta(\alpha_1 \tilde{K}_{\alpha_2\cdots\alpha_p})
\]

(1)

**Definition**

A totally antisymmetric tensor is a conformal Killing-Yano tensor if it obeys the following equation:

\[
\nabla(\alpha_1 h_{\alpha_2})\alpha_3\cdots\alpha_{p+1} = g_{\alpha_1\alpha_2} \tilde{h}_{\alpha_3\cdots\alpha_{p+1}} - (p - 1)g[\alpha_3(\alpha_1 \tilde{h}_{\alpha_2})\cdots\alpha_{p+1}]
\]

(2)
Generalized Killing-Yano and Stäckel-Killing tensors and PCKY

Definition

A PCKY tensor is a principal conformal Killing-Yano tensor, i.e. a rank 2 conformal Killing-Yano tensor, which generates towers of Killing-Yano and Stäckel-Killing tensors of various ranks on manifolds of various dimensions.

To be noted the fact that the same definitions for Killing-Yano and Stäckel-Killing tensors are valid in the cases where torsion is present, but some of the properties of these tensors differ in the case with torsion, for instance: if a classical symmetry transfers into a quantum symmetry with no anomalies in the case with no torsion, the anomalies may be present in the case with torsion.
Killing spinors

The equation of symplectic Majorana 5-dimensional Killing spinors is:

\[
D_\mu \epsilon_i = iM_{ij} \frac{a}{2} \gamma_\mu \epsilon_j \tag{3}
\]

where

\[
M = \vec{x} \vec{\sigma} \text{ with } \vec{\sigma} \text{ the Pauli matrices and}
\]

\[
\vec{x} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \tag{4}
\]

If a Killing spinor \( \psi \) exists for a (pseudo) Riemannian manifold then a Killing-Yano tensor of rank-\( p \) \( \omega_p \) exists and it can be constructed as:

\[
\omega_p(X_1 \cdots X_p) = \langle [X_1 \wedge \cdots \wedge X_p] \vee \psi, \psi \rangle \tag{5}
\]
Internal symmetries of minimal gauged supergravities in 5 dimensions and the Killing-Maxwell system
The minimal gauged supergravities spacetimes in the view of an artist...

Figure: ©Violeta Dobra, adapt.ro
The solutions of this type are classified in two classes depending whether the Killing vector is null or timelike.

- The timelike solution is: a manifold with a 4-dimensional Kähler base with a U(2) structure which maintains 1/4 of the supersymmetry.

- The null solution: it maintains 1/4 of the supersymmetry and is fixed up to 3 harmonic functions.
The minimal gauged supergravities in 5 dimensions are closely related to Kerr-Newman spacetimes.

The minimal gauged supergravities in 5 dimensions are endowed with torsion and the Cher-Simons term $*F$ plays this role.

The PGCKY (the generalized principal conformal Killing-Yano tensor) of the minimal gauged supergravities is given by the equation:

$$\nabla_\rho h_{\mu\nu} = 2g_\rho[\mu \xi_\nu] - \frac{1}{\sqrt{3}}(\ast F)_\rho^\sigma[\mu h^\sigma_\nu].$$  \hspace{1cm} (6)
The Killing-Maxwell system for minimal gauged supergravities

This system was introduced back in 1987 by Brandon Carter for Kerr-Newman spacetimes and is determined by the following condition on the electromagnetic potential, which holds also for 5 dimensional minimal gauged supergravities:

\[ \hat{A}_{[\mu;\nu];\rho} = 2 \frac{4\pi}{3} \hat{j}_{[\mu}g_{\nu]\rho}, \]  

(7)

where \( j \) is the associated current, which must be a Killing vector and \( g \) is the metric of the spacetime.

Naturally the Maxwell equations hold and so we get the Killing-Maxwell system:

\[ \hat{F}_{\rho \mu} = 4\pi \hat{j}^\mu \]  

(8)

\[ \hat{F}_{[\mu\nu;\rho]} = 0. \]  

(9)
The Killing-Maxwell system with Killing-Yano torsion

In the case that the torsion is a Killing-Yano tensor the equation (6) becomes:

$$\nabla_\rho h_{\mu\nu} = 2g_\rho[\mu \xi_\nu].$$  (10)

If in the above equation we make $j = \pi \xi$ we obtain a Killing-Maxwell system with the Maxwell field notated by $h$, being the PGCKY of the spacetime (which generates towers of Killing-Yano and Stäckel-Killing tensors) and $h$ is a Killing-Yano tensor.
The symmetries of the Chong-Cvetič-Lü-Pope black hole
The Chong-Cvetič-Lü-Pope black hole in the view of an artist...

Figure: ©Violeta Dobra, adapt.ro
The Chong-Cvetič-Lü-Pope black hole

- it is a $D=5$ minimal supergravity solution, a rotating, charged non-extremal solution
- it is endowed with torsion: the Chern-Simons 2-form, $*F$, is assimilated with Killing-Yano torsion
- it is a solution characterized by four parameters: mass, charge, and 2 independent rotation parameters
- in Boyer-Lindquist coordinates this solution of $D=5$ minimal gauged supergravity is static (non-rotating) asymptotically
The Chong-Cvetič-Lü-Pope black hole metric

The metric is given by:

\[
g = \sum_{\mu=x,y} (\omega^\mu \omega^\mu + \tilde{\omega}^\mu \tilde{\omega}^\mu) + \omega^\epsilon \omega^\epsilon, \tag{11}\]

\[
A = \sqrt{3}(A_q + A_p). \tag{12}\]

And-

\[
\omega^x = \sqrt{\frac{x - y}{4X}} \, dx, \quad \tilde{\omega}^x = \frac{\sqrt{X}(dt + yd\phi)}{\sqrt{x(y - x)}}, \tag{13}\]

\[
\omega^y = \sqrt{\frac{y - x}{4Y}} \, dy, \quad \tilde{\omega}^y = \frac{\sqrt{Y}(dt + xd\phi)}{\sqrt{y(x - y)}}, \tag{14}\]
The metric

\[
\omega^\epsilon = \frac{1}{\sqrt{-xy}} [\mu dt + \mu (x + y) d\phi + xy d\psi - y A_q - x A_p],
\]

(15)

\[
A_q = \frac{q}{x - y} (dt + y d\phi), \quad A_p = \frac{-p}{x - y} (dt + x d\phi),
\]

(16)

and

\[
X = (\mu + q)^2 + A x + C X^2 + \frac{1}{12} \Lambda x^3,
\]

(17)

\[
Y = (\mu + p)^2 + B y + C y^2 + \frac{1}{12} \Lambda y^3.
\]

(18)
The generalized conformal Killing-Yano and Stäckel-Killing tensors of the spacetime

\[ F = \sqrt{-x\tilde{\omega}^x} \wedge \omega^x + \sqrt{-y\tilde{\omega}^y} \wedge \omega^y \]  

(19)

\[ K^{(F)} = y(\omega^x \omega^x + \tilde{\omega}^x \tilde{\omega}^x) + x(\omega^y \omega^y + \tilde{\omega}^y \tilde{\omega}^y) + (x + y)\omega^\epsilon \omega^\epsilon. \]  

(20)

\[ h^{(\psi)}_{ab} = 4\omega[a(\partial_\psi)b] \]  

(21)

\[ K^{\psi}_{ab} = 16\omega[a(\partial_\psi)c]\omega[b(\partial_\psi)c] - 4g_{ab}(\omega^d \omega^c (\partial_\psi)_c(\partial_\psi)^d - \omega^d \omega^c (\partial_\psi)_c(\partial_\psi)^d), \]  

(22)

\[ h^{(\phi)}_{ab} = 4\omega[a(\partial_\phi)b]. \]  

(23)

\[ K^{\phi}_{ab} = 16\omega[a(\partial_\phi)c]\omega[b(\partial_\phi)c] - 4g_{ab}(\omega^d \omega^c (\partial_\phi)_c(\partial_\phi)^d - \omega^d \omega^c (\partial_\phi)_c(\partial_\phi)^d). \]  

(24)
Dirac-type operators for this spacetime with Killing-Yano torsion

\[ \hat{Q}_Y^A = \gamma^\mu Y_\mu^\nu D^A_\nu - \frac{1}{6} \gamma^\mu \gamma^\nu \gamma^\rho \nabla_\mu Y_{\nu\rho}. \] (25)

with

\[ \{ \hat{Q}_Y^A, D^A \} = 0 \] (26)

Hence there are no gravitational anomalies in this case. Same for the higher rank operators:

\[ \hat{Q}_Y^{A,p} = \gamma^{\mu_1} \cdots \gamma^{\mu_{p-1}} Y_{\mu_1 \cdots \mu_{p-1}}^\nu D^A_\nu - \frac{(-1)^p}{2(p + 1)} \gamma^\nu \gamma^{\mu_1} \cdots \gamma^{\mu_p} \nabla_\nu Y_{\mu_1 \cdots \mu_p}. \] (27)
The black hole solution Houri - Takeuchi - Yasui
The Houri-Takeuchi-Yasui spacetime in the view of an artist ...

Figure: ©Violeta Dobra, adapt.ro
The Houri-Takeuchi-Yasui metric

\[ g = (\xi - x)(d\theta^2 + \sin^2\theta d\phi^2) + \frac{dx^2}{Q(x)} + Q(x)(d\psi_1 + \cos\theta d\phi)^2 + 
\]

\[ + 4(d\psi_0 + (x + \frac{q}{x - \xi})d\psi_1 + (x - \xi + \frac{q}{x - \xi})\cos\theta d\phi)^2 \quad (28) \]

where

\[ Q(x) = \frac{4x^3 + (1 - 12\xi)x^2 + (8q - 2\xi + 12\xi^2)x + k}{\xi - x} \quad (29) \]

and q, \(\xi\) and k are free parameters.
The action for this metric and the equations of motion

\[ \mathcal{L}_5 = *(\mathcal{R} - \Lambda) - \frac{1}{2} F \wedge \ast F + \frac{1}{3 \sqrt{3}} F \wedge F \wedge A \]  

(30)

where \( F = dA \) is the Maxwell potential:

\[ A = -\frac{2\sqrt{3}q}{x - \xi} (d\psi_1 + \cos \theta d\phi) \]  

(31)

and the torsion is: \( T = \ast F / \sqrt{3} \). The equations of motion are:

\[ R_{ab} = -4g_{ab} + \frac{1}{2} (F_{ac} F_b^c - \frac{1}{6} g_{ab} F_{cd} F^{cd}) \]  

(32)

\[ d \ast F - \frac{1}{\sqrt{3}} F \wedge F = 0 \]  

(33)
As part of the performed calculations we obtained:

- the inverse metric,
- the associated Riemann tensor,
- the Christoffel symbols,
- the spin connections of this metric.
The Killing spinor equation and its solution for the Houri-Takeuchi-Yasui metric

\[ D_\alpha + \frac{1}{4\sqrt{3}}(\gamma_\alpha^{\beta\gamma} - 4\delta_\alpha^{\beta\gamma})F_{\beta\gamma}]\epsilon^{a} - \chi\epsilon^{ab}(\frac{1}{4\sqrt{3}}\gamma_\alpha - \frac{1}{2}A_\alpha)\epsilon^{b} = 0, \quad (34) \]

An untrivial result is the solution of the above equation given the metric:

\[ \epsilon_{i} = (e_{\frac{i}{2}}\gamma^{i}x^{M})_{j}^{k}(\delta_{i}^{j}\chi\gamma^{\alpha}(\gamma_{\alpha}^{\beta\delta} - \delta_{\alpha}^{\beta\gamma})F_{\beta\delta} + \frac{i\epsilon^{jl}}{2}\chi\gamma^{\alpha}(M_{il} - i\delta_{il}A_\alpha\gamma^{\alpha}))\xi_{k} \quad (35) \]

where \( M = \vec{x}\vec{\sigma} \) with \( \vec{\sigma} \) are the Pauli matrices and

\[ \vec{x} = (\sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta) \quad (36) \]

and \( \epsilon^{jki} \) and \( \epsilon^{ij} \) are the Levi-Civita tensors and \( \xi_{k} \) is a Majorana symplectic spinor.
The integrability conditions for the Killing spinors

\[
\left\{ \frac{1}{8} R_{\rho\mu\nu_1\nu_2} \gamma^{\nu_1\nu_2} + \frac{1}{4\sqrt{3}} (\gamma_{[\mu}^{\nu_1\nu_2} + 4 \gamma^{\nu_1} \delta_{[\mu}^{\nu_2}) \nabla_\rho] F_{\nu_1\nu_2} - \right. \\
\left. - \frac{1}{48} (-2 F^2 \gamma_{[\rho}^{\mu}] - 8 F_{[\rho}^{\nu} \gamma_{\mu]}^{\nu} + 12 F_{\mu\nu_1} F_{\rho\nu_2} \gamma^{\nu_1\nu_2} + 8 F_{\nu_1\nu_2} F_{\nu_3[\rho} \gamma_{\mu]}^{\nu_1\nu_2\nu_3}) - \\
- \frac{\chi^2}{48} \gamma_{[\rho\mu]} \epsilon^a - \frac{\chi}{24} (\gamma_{[\rho\mu}^{\nu_1\nu_2} F_{\nu_1\nu_2} - 4 F_{[\rho}^{\nu} \gamma_{\mu]}^{\nu} - F_{\rho\mu}] \epsilon^{ab} \epsilon^b = 0 \right. 
\] (37)
The equivalence of the conformal Killing-Yano tensors and Killing spinors for arbitrary dimensional Sasaki-Einstein manifolds
The definition of Sasaki-Einstein manifolds

Definition
A compact Riemannian manifold \((S,g)\) is a Sasaki manifold iff the metric of the associated cone \((C(S) = \mathcal{R}_{>0}XS, \bar{g} = dr^2 + r^2g)\) is Kähler.

Definition
A Sasaki-Einstein manifold is a Sasaki manifold \((S,g)\) with \(Ric_g = 2(n - 1)g\).
The equivalence of the conformal Killing-Yano tensors with Killing spinors for Sasaki-Einstein manifolds

Lemma

For an arbitrary dimensional Sasaki-Einstein manifold, if a Killing-Yano tensor exists, then a Killing spinor exists as well.

Remarci

- If a tensor is Killing-Yano, then it is also a conformal Killing-Yano tensor.
- If there exists a Killing spinor on a manifold, then there exists also a conformal Killing-Yano tensor on that manifold.
Conclusions

- We have presented some definitions and properties of Stäckel-Killing and Killing-Yano tensors, and Killing spinors.

- We have presented some properties and known results about minimal gauged supergravities and we have introduced the Killing-Maxwell system, by analogy with the Kerr-Newman spacetime.

- We have presented some known results about the Chong-Cvetič-Lü-Pope black hole as an example of minimal gauged supergravity in 5 dimensions together with some new results (Killing-Yano and Stäckel-Killing tensors) for this spacetime.

- We found the Killing spinors for the Houri-Takeuchi-Yasui metric, an example of minimal gauged supergravity with Sasaki structure deformed by torsion.

- We have proved the equivalence of the conformal Killing-Yano tensors and Killing spinors for Sasaki-Einstein manifolds.