Earth matter effects in long baseline neutrino experiments

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HOW MUCH EARTH MATTER ON THE WAY

DEPTH AT DISTANCE 'X' FROM SOURCE

\[ r^2 = \frac{(L/2 - x)^2}{h^2} \quad \text{and} \quad h^2 = R^2 - \frac{(L/2)^2}{d = R - r = R - \sqrt{R^2 + x^2 - Lx}} \]
PRELIMINARY REFERENCE EARTH MODEL
Simple case in vacuum

1. \( H\psi = E\psi \rightarrow i\partial\psi/\partial t = E\psi \rightarrow \psi(t) = e^{-iEt} \psi(0) \)

2. Flavor \( \rightarrow \) Greek, Mass \( \rightarrow \) Roman

3. \( \psi_\alpha = \sum_i U_{\alpha i} \psi_i \)

4. \( \nu_\alpha(t) = \sum_\beta [\sum_i U_{\alpha i} e^{-iE_i t} U^*_{\beta i}] \nu_\beta \)

5. \( P(\alpha \rightarrow \beta) \equiv P_{\beta \alpha} = \sum_{ij} U_{\alpha i}^* U_{\alpha j} U_{\beta i}^* U_{\beta j} \ e^{i\Delta_{ij}} \) \( \Delta_{ij} = \Delta m^2_{ij} \frac{L}{2E} \)

6. Expanding \( e^{i\Delta_{ij}} \) in terms of \( \sin \Delta_{ij} \) and \( \cos \Delta_{ij} \) we obtain

\[
P_{\beta \alpha} = \delta_{\beta \alpha} - 4 \sum_{i>j} Re(w) \sin^2 \frac{\Delta_{ij}}{2} + 2 \sum_{i>j} Im(w) \sin \Delta_{ij}
\]

\[
w_{\beta \alpha}^{ij} = U_{\alpha i} U_{\alpha j}^* U_{\beta i}^* U_{\beta j}
\]

7. \( Im(w) \) changes sign for \( U \rightarrow U^* \) but \( Re(w) \) remains unchanged.

\[
P_{\beta \alpha} = P_{\beta \alpha}^{CP-odd} + P_{\beta \alpha}^{CP-even}
\]

8. Then we can ‘define’ an asymmetry

\[
A_{\beta \alpha} = \frac{P(\alpha \rightarrow \beta) - P(\bar{\alpha} \rightarrow \bar{\beta})}{P(\alpha \rightarrow \beta) + P(\bar{\alpha} \rightarrow \bar{\beta})} = \frac{P_{\alpha \beta}^{CP-odd}}{P_{\alpha \beta}^{CP-even}} = \frac{P_{\alpha \beta}^{CP-odd}}{\text{Normalization}}
\]

9. Note that one can ‘parametrize’ \( U \) in many ways. In each parametrization one will have ‘three’ angles and one ‘phase’
1. Let us consider the time evolution of a neutrino flavor state

\[ |\nu_i > = e^{-iHt}|\nu_0 >, \]

\[ |\nu(x) > = e^{-ix\hbar}\nu(0) > \quad (\hbar = c = 1, \quad x = ct). \]

for any \( x \). In fact, when the perturbation \( H' \) is \( x \)-dependent (and the only source of such a dependence is spatial variation in the earth's matter density function), then \( H = H_0 + H' \) and Eqn (4) is no longer true. Instead, we have to write,

\[ |\nu(x) > = Te^{-i \int_0^x ds H(s)}|\nu(0) >. \]

Note that there is no harm in keeping the \( T \) symbol.

2. Numerical Integration

(a) In \( t \) seconds neutrino traverses a distance \( L \)

(b) Let us note that \( \nu^m(t) = e^{-\int_0^L H(s) ds} \nu^m(0) = \kappa(L) \nu^m(0) \)

(c) \( \nu^f = U\nu^m \)

(d) \( \nu^f(t) = U\nu^m(t) \)

(e) \( \nu^f(t) = [U\kappa(L)U^{-1}] \nu^f(0) \)

3. Numerical Differential Equation

\[ |\nu(x) > = S(x)|\nu(0) >, \]

\[ i\frac{dS(x)}{dx} = H(x)S(x). \]
FLOW CHART OF CALCULATIONS

1. PERTURBATION THEORY

\[ |\nu(x)\rangle = S(x)|\nu(0)\rangle, \]  
\[ S(x) = T e^{-i \int_0^x dy H(y)}, \]  
\[ S(0) = 1. \]  

2. HAMILTONIAN IS DIVIDED INTO MAIN PART AND PERTURBATION PART

\[ H(y) = H_0 + H_1(y), \]  
\[ H_0 = \frac{1}{2E} U \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \delta m_{31}^2 \end{pmatrix} U^\dagger, \]  
\[ H_1 = \frac{1}{2E} U \begin{pmatrix} 0 & 0 & 0 \\ 0 & \delta m_{21}^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} U^\dagger + \begin{pmatrix} a(y) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \]  

3. RELATIVE MAGNITUDES IN HAMILTONIAN ARE

\[ \delta m_{21}^2 \ll \delta m_{31}^2, \]  
\[ a(y) \ll \delta m_{31}^2. \]  

4. IDEA IS TO SOLVE FOR \( S(x) \) AND FIND SURVIVAL AND TRANSITION PROBABILITIES USING

\[ P(x) = S^\dagger(x) S(x). \]  

THIS WILL GIVE \( 3 \times 3 = 9 \) PROBABILITIES FOR THREE GENERATIONS.

5. WHILE SQUAREING \( S(x) \) WE WILL KEEP LINEAR TERMS IN \( \delta m_{21}^2 \) AND IN \( a(y) \).
SOME RELEVANT STEPS IN OUR CALCULATION (SEE: B.B., CHOUBEY AND ROY)

SOLUTION FOR $S(x)$ IS

$$S(x) = e^{-iH_0x} \left(1 - i \int_0^x ds \ e^{iH_0s} H_1(s) \ e^{-iH_0s}\right). \quad (15)$$

THE SOLUTION HAS TWO ADDITIVE FACTORS

$$S(x) = S_0(x) + S_1(x). \quad (16)$$

FIRST PART DEPENDS ON MAIN HAMILTONIAN AND SECOND PART DEPENDS ON PERTURBATION PARTS

$$S_0(x) = e^{-\left(i/2E\right) U \begin{pmatrix} 0 & 0 \\ 0 & \Delta m_{31}^2 \end{pmatrix} U^\dagger x} \quad (18)$$

$$S_1(x) = -i \int_0^x ds \ e^{-iH_0(x-s)} H_1(s) e^{-iH_0s}. \quad (19)$$

LET US FIRST SIMPLIFY $S_0$ AND RENAME IT $A$

$$A_{\beta\alpha} = S_0^0(x)_{\beta\alpha} = \delta_{\beta\alpha} - 2i \ U_{\beta 3} \ U_{\alpha 3}^* \ e^{-i \Delta m_{31}^2 x / 4E} \ \sin \frac{\Delta m_{31}^2 x}{4E} \quad (20)$$

BECAUSE PERTURBATION HAMILTONIAN HAS TWO PARTS

(a) SOLAR OSCILLATION PART
(b) EARTH MATTER PART

$S_1(x)$ CAN BE DECOMPOSED INTO TWO ADDITIVE PARTS AGAIN.
PART (a) IS PROPORTIONAL TO $\Delta m_{21}^2$

$$S^1(x)_{\beta\alpha} = -i \frac{\Delta m_{21}^2}{2E} x U_{\beta 2} U^*_{\alpha 2}$$
$$= -i B_{\beta\alpha} \Delta m_{21}^2, \quad (21)$$

PART (b) DEPENDS ON THE EARTH MATTER PROFILE

$$S^a(x)_{\beta\alpha} = -i \frac{\Delta m_{31}^2}{2E} U_{\beta i} U^*_{\alpha j} U^i_{11} U^j_{11} f^a_{ij}(x). \quad (22)$$

Thus $f^a(x)$ is the “MATTER FUNCTION”,

$$f^a_{ij}(x) = \delta_{i3} \delta_{j3} e^{-i \frac{\Delta m_{31}^2 x}{2E}} \int_0^x ds \ a(s,E)$$
$$+(1-\delta_{i3}) (1-\delta_{j3}) \int_0^x ds \ a(s,E)$$
$$+(1-\delta_{i3}) \delta_{j3} \int_0^x ds \ a(s,E) e^{-i \frac{\Delta m_{31}^2 s}{2E}}$$
$$+\delta_{i3}(1-\delta_{j3}) \int_0^x ds \ a(s,E) e^{-i \frac{\Delta m_{31}^2 (x-s)}{2E}}. \quad (23)$$

IT BECOMES EASY IF WE ASSUME A CONSTANT MATTER PROFILE HERE. BUT WE WILL WORK WITH A GENERAL MATTER PROFILE.

$$S_{\beta\alpha}(x/E) = A_{\beta\alpha}(x/E) - i B_{\beta\alpha}(x/E) \Delta m_{21}^2 + S^a_{\beta\alpha}(x). \quad (24)$$

$$P_{\beta\alpha} = A^*_{\beta\alpha} A_{\beta\alpha} - 2 \text{Im}(B^*_{\beta\alpha} A_{\beta\alpha}) \Delta m_{21}^2 + 2 \text{Re}[A_{\beta\alpha} S^a_{\beta\alpha}]. \quad (25)$$
A SIMPLE EXAMPLE: CALCULATION OF THE PROBABILITY $P_{\mu\mu}$

CONSIDER THE CASE $\alpha = \mu, \beta = \mu$. IN THIS CASE $A, B, S^\alpha$ BECOMES,

\begin{align*}
A_{\mu\mu}(x/E) &= 1 - 2i |U_{\mu3}|^2 e^{-ig(x/E)} \sin[g(x/E)], \quad (26) \\
B_{\mu\mu}(x/E) &= |U_{\mu2}|^2 \frac{x}{2E}, \quad (27) \\
S_{\mu\mu}^\alpha(x, E) &= -i \frac{|U_{e3}|^2 |U_{\mu3}|^2}{2E} M(x, E), \quad (28)
\end{align*}

$M$ IS THE CRUCIAL INTEGRAL CONTAINING EARTH MATTER EFFECTS

\begin{align*}
M(x, E) &= e^{-ig(x/E)} R(x, E), \quad (29) \\
R(x, E) &= 2 \int_0^x ds \ a(s, E) \cos g(x/E) - \cos \{g(x/E) - k(s/E)\}. \quad (30)
\end{align*}

$g$ and $k$ ARE TWO FUNCTIONS DEFINED AS,

\begin{align*}
g(x/E) &\equiv \frac{\Delta m_{31}^2 x}{4E}, \quad (32) \\
k(s/E) &\equiv \frac{\Delta m_{31}^2 s}{2E}. \quad (33)
\end{align*}
\[ P_{\mu\mu}(L, E) = P^0_{\mu\mu}(L/E) + P^a_{\mu\mu}(L, E), \]  

where
\[ P^a_{\mu\mu}(L, E) = \frac{|U_{e3}|^2|U_{\mu3}|^2}{E} (2|U_{\mu3}|^2 - 1) \sin[g(L/E)] R(L, E). \]

The asymmetry \( \Delta P_{\mu\mu}(L, E) \), defined as
\[ \Delta P_{\beta\alpha}(L, E) = P[\nu_\alpha(0) \to \nu_\beta(L)] - P[\bar{\nu}_\alpha(0) \to \bar{\nu}_\beta(L)] \]  

\[ \Delta P_{\mu\mu}(L, E) = \frac{|U_{e3}|^2|U_{\mu3}|^2}{E} (2|U_{\mu3}|^2 - 1) \sin[g(L/E)] [R^+(L, E) - R^-(L, E)] \]
\[ = \frac{2|U_{e3}|^2|U_{\mu3}|^2}{E} (2|U_{\mu3}|^2 - 1) \sin[g(L/E)] R^+(L, E). \]

In (37) \( R^\pm(L, E) \) is obtained by using \( a(s, E) \equiv \pm|a(s, E)| \) respectively so that \( R^-(L, E) = -R^+(L, E) \). Thus we see that the matter effect contribution to the muonic flavour survival probability \( P_{\mu\mu}(x) \) vanishes if \( |U_{\mu3}| = 1/\sqrt{2} \). Moreover, even in matter, any nonzero asymmetry \( \Delta P_{\mu\mu} \), detected from the muon (anti) neutrino flavour survival probabilities will signal a deviation from the condition for the strictly maximal mixing of atmospheric neutrinos at super-K. This can be used in future to sensitively probe any deviation of \( |U_{\mu3}| \) from its maximal value \( 1/\sqrt{2} \).
$\alpha = \beta \neq e$

In this case the matter independent and matter dependent transition probabilities

$$
P^0_{\alpha \alpha}(L/E) = 4|U_{\alpha 3}|^2(1 - |U_{\alpha 3}|^2)\sin^2[g(L/E)]
+ \frac{\Delta m^2_{31} L}{E}|U_{\alpha 2}|^2|U_{\alpha 3}|^2\sin[2g(L/E)]
$$

(38)

$$
P^a_{\alpha \alpha}(L, E) = \frac{1}{E}|U_{\alpha 3}|^2|U_{e 3}|^2(2|U_{\alpha 3}|^2 - 1)\sin[g(L/E)]R(L, E)
$$

(39)

$\alpha = \beta = e$,

$P^0$ remains the same but $P^a$ changes.

$$
P^a_{ee}(L, E) = \frac{1}{E}|U_{e 3}|^2\sin^2[g(L/E)] \int_0^L ds \; a(s, E)\sin[g(L/E)]
-k(s/E)/2] - \frac{1}{E}|U_{e 3}|^4 \sin[g(L/E)]R(L, E)
+ \frac{2}{E}|U_{e 3}|^4(|U_{e 3}|^2 - 1)\sin[g(L/E)]R(L, E).
$$

(40)
\[\alpha \neq \beta \text{ CASES CAN BE SUMMARIZED AS WELL AS FOLLOWS}\]

\[P_{\beta\alpha}^0 = 4|U_{\beta 3}|^2|U_{\alpha 3}|^2 \sin^2[g(L/E)]\]

\[\frac{2\Delta m_{21}^2 L}{E} \left\{ Im\xi \sin^2[g(L/E)] \frac{1}{2} Re\xi \sin[2g(L/E)] \right\} \] (41)

\[\xi \equiv U_{\beta 2}^* U_{\beta 3} U_{\alpha 2} U_{\alpha 3}^*. \] (42)

- \(\alpha \neq e, \beta \neq e\)

\[P_{\beta\alpha}^a (L, E) = \frac{2}{E}|U_{\alpha 3}|^2|U_{\beta 3}|^2|U_{e 3}|^2 \sin[g(L/E)]R(L, E)\] (43)

- \(\alpha \neq e, \beta = e\)

\[P_{e\alpha}^a (L, E) = \frac{1}{E}|U_{\alpha 3}|^2|U_{e 3}|^2 (2|U_{e 3}|^2 - 1) \sin[g(L/E)]R(L, E). \] (44)

- \(\alpha = e, \beta \neq e\)

\[P_{\beta e}^a (L, E) = \frac{1}{E}|U_{e 3}|^2|U_{\beta 3}|^2 (2|U_{e 3}|^2 - 1) \sin[g(L/E)]R(L, E). \] (45)
CONCLUSIONS

1. WE HAVE DONE A PERTURBATION THEORY CALCULATION

2. WE HAVE CONSIDERED A GENERAL MATTER PROFILE. THEREFORE THESE RESULTS ARE INDEPENDENT OF ANY SPECIFIC GEOLOGICAL MODEL.

3. WE HAVE NOT RESTRICTED OURSELVES TO ANY SPECIFIC PARAMETRIZATION OF THE LEPTON MIXING MATRIX $U_{ij}$.

4. WE HAVE GIVEN COMPACT FORMULAS FOR $\nu_\mu \leftrightarrow \nu_\mu$ CASE AND SHOWN THAT MATTER DEPENDENT ASYMMETRY WILL VANISH FOR STRICTLY MAXIMAL MIXING. THIS RESULT IS INDEPENDENT OF NEUTRINO ENERGY, BASELINE LENGTH, U MATRIX PARAMETRIZATION AND EARTH’S DENSITY PROFILE.

5. WE HAVE DERIVED ANALYTICAL FORMULAS FOR MATTER INDUCED TERMS IN GENERAL TRANSITION AND SURVIVAL PROBABILITIES. THESE EXTRA EFFECTS HAVE TO BE SUBTRACTED FROM TOTAL TRANSITION AND SURVIVAL PROBABILITIES TO UNCOVER TRUE PROBABILITIES IN VACUUM.