Restriction theorems for higher-dimensional curves (part 3)

Jong-Guk Bak

POSTECH

Workshop on harmonic analysis, National Central University, Taiwan, July 2010
Joint work with

- Dan Oberlin and Andreas Seeger
The Fourier restriction operator and its adjoint

Let \( \gamma(t) \) be a suitable curve in \( \mathbb{R}^d \), and let \( \mathcal{R}f(t) = \hat{f}(\gamma(t)) \) denote the (Fourier) restriction operator.

\[
\mathcal{R}f(t) = \hat{f}(\gamma(t)) = \int_{\mathbb{R}^d} e^{-ix \cdot \gamma(t)} f(x) dx.
\]

Let us consider the restriction estimate

\[
\|\mathcal{R}f\|_{L^q(I)} \leq C \|f\|_{L^p(\mathbb{R}^d)} \tag{1}
\]

where \( I \) is an interval, finite or infinite.

Then the adjoint \( T = \mathcal{R}^* \), called the Fourier extension operator, is given by

\[
Tf(x) = \int_{I} e^{ix \cdot \gamma(t)} f(t) dt. \tag{2}
\]

And the dual estimate of (1) is given by

\[
\|Tf\|_{L^{p'}(\mathbb{R}^d)} \leq C \|f\|_{L^{q'}(I)} \tag{3}
\]

where \( p' \) is the conjugate exponent of \( p \), i.e. \( 1/p + 1/p' = 1 \).
For the parabola $\gamma(t) = (t, t^2)$ (or the circle), the following holds:

$$\left( \int I|\hat{f}(t, t^2)|^q dt \right)^{1/q} \leq C\|f\|_{L^p(R^2)}, \quad \frac{1}{q} = 3 \left( 1 - \frac{1}{p} \right) \quad (4)$$


However, the endpoint case of (4), i.e. for $p = 4/3$, fails.

Moreover, (4) fails even for characteristic functions $f$ when $p = 4/3$. That is, the weaker substitute $L^{4/3,1}(R^2)$-$L^q$ estimate fails as well (for any $q < \infty$). (A Kakeya set counterexample has been found by Beckner-Carbery-Semmes-Soria 1989.)
A nondegenerate curve in $\mathbb{R}^d$, $d \geq 3$

Let

$$\gamma(t) = (t, t^2, \cdots, t^d), \quad t \in \mathbb{R}. \quad (5)$$

Prestini 1979, Christ 1985: some partial results.

Drury (1985) obtained the full result (range) by the method of “offspring curves” (an inductive argument):

$$\left( \int |\hat{f}(\gamma(t))|^q dt \right)^{1/q} \leq C \|f\|_p, \quad \frac{1}{q} = \frac{d(d + 1)}{2} \left(1 - \frac{1}{p}\right),$$

for $1 \leq p < p_d := (d^2 + d + 2)/(d^2 + d)$. (When $d = 3$, we have $p' = 6q$, $p_3 = 7/6$.)
An endpoint estimate for $p = p_d$, $d \geq 3$

- Endpoint $L^p - L^q$ estimate fails for $p = p_d$.
  (Arkhipov-Chubarikov-Karachuba 1979)

- However, somewhat surprisingly it turns out that the endpoint $L^{p_d, 1} - L^{p_d}$ estimate (restricted strong type) does hold when $d \geq 3$:

$$\left( \int |\hat{f}(\gamma(t))|^{p_d} dt \right)^{1/p_d} \leq C \|f\|_{L^{p_d, 1}(\mathbb{R}^d)}$$  \hspace{1cm} (7)

- In fact a dual version of this has been shown also for a general nondegenerate phase $\phi(x, t)$. (Oberlin-Seeger-B 2009)
An estimate for $T$: the dual estimate

For $d \geq 3$, $T$ is of strong type $(p, q)$, if $1 \leq p < q_d$, $1/q = [2/(d^2 + d)](1 - 1/p)$:

$$\|Tf\|_{L^q(\mathbb{R}^d)} \leq C\|f\|_{L^p(\mathbb{R})}$$  \hspace{1cm} (8)

where $q_d = p' = (d^2 + d + 2)/2$. (For $d = 3$, $q_d = 7$.)

The estimate (8) follows from the endpoint result: $T$ is weak type $(p, q)$ for $p = q = q_d$. (Oberlin-Seeger-B 2009)
Figure: The critical line for the adjoint operator $T$ in $\mathbb{R}^3$. 
An operator with a general nondegenerate phase

\[ T_\lambda f(x) = \int_I e^{i\lambda \phi(x,t)} a(x, t) f(t) dt \quad (9) \]

where \( \lambda > 1 \), the amplitude \( a \) is \( C^\infty \) and compactly supported in \( \mathbb{R}^d \times \mathbb{R} \), and \( \phi \) is a real-valued phase function in \( C^\infty(\Omega \times I) \). For example, \( \phi(x, t) = x \cdot \gamma(t) \).

The nondegeneracy condition

\[ \det (\partial_t(\nabla_x \phi), \partial^2_t(\nabla_x \phi), \ldots, \partial^d_t(\nabla_x \phi)) \neq 0. \quad (10) \]

(a generalization of a condition of Hörmander for \( d = 2 \): Lee-B 2004)

For \( d \geq 3 \) and \( \lambda > 1 \),

\[ \| T_\lambda f \|_{L^q(\mathbb{R}^d)} \leq C \lambda^{-d/q} \| f \|_{L^p(\mathbb{R}^d)} \quad (11) \]

for the full range of exponents: \( 1 \leq p < q_d \) on the critical line. (Lee-B 2004)

The endpoint result:

\[ \| T_\lambda f \|_{L^{q_d,\infty}(\mathbb{R}^d)} \leq C \lambda^{-d/q_d} \| f \|_{L^{q_d}(\mathbb{R}^d)} \quad (12) \]
Degenerate curves in $\mathbb{R}^2$

- A “damped” estimate for convex degenerate curves

$$\left( \int |\hat{f}(\gamma(t))|^q w(t) dt \right)^{1/q} \leq C \| f \|_{L^p(\mathbb{R}^2)}, \quad q = p'/3, \quad (13)$$

for all $p < 4/3$. (Sjölin 1974)

- Here the weight function (or damping factor) is $w(t) = |\det(\gamma'(t), \gamma''(t))|^{1/3}$. It mitigates (compensates for) the effect of vanishing curvature at some points. (The measure $wdt$ is called the “affine arclength measure”.)

- For example, $\gamma(t) = (t, t^k), \ k > 2$, and then $w(t) = c_k t^{(k-2)/3}, \ t > 0$. 
Degenerate curves in $\mathbb{R}^d$

▶ Question: does the following estimate hold?

$$\left( \int_I |\hat{f}(\gamma(t))|^q w(t) dt \right)^{1/q} \leq C \|f\|_{L^p(\mathbb{R}^d)}$$

(14)

for $1/q = d(d + 1)/(2p')$, $1 \leq p < p_d$.

▶ Here $w(t) = |\tau(t)|^{2/(d^2 + d)}$, and $\tau(t)$ is the “torsion” defined by

$$\tau(t) = \det(\gamma'(t), \cdots, \gamma^{(d)}(t)).$$

▶ Drury-Marshall 1985, 1987; Drury 1990: obtained some partial results for $1 \leq p < p_*$ (for some $p_* < p_d$) for some special cases.
For example, consider $\gamma(t) = (t, t^2, t^\beta)$, $d = 3$. The only three cases for which the restriction estimate was known for the full range $1 \leq p < 7/6$ were:

- (i) $\beta = 2$ (the trivial case with $w(t) = 0$),
- (ii) $\beta = 3$ (the nondegenerate case: Drury 1985),
- (iii) $\beta = 9$ (the “rough” nondegenerate case) Oberlin-B 2003.

Some new results:

- The full range $1 \leq p < p_d$ has been obtained for $d \geq 3$ for some classes of curves. (Oberlin-Seeger-B, 2008, 2009)
- In addition, the endpoint results (restricted strong type for $p = p_d$, $d \geq 3$) are also known now. (Oberlin-Seeger-B, in preparation)
The known cases so far, in the full range of $p$

- (i) the curves of “simple” type whose last component is an arbitrary polynomial

$$\gamma(t) = \gamma_b(t) = \left(t, \frac{t^2}{2!}, \ldots, \frac{t^{d-1}}{(d-1)!}, \phi(t)\right), \quad t \in \mathbb{R}, \quad (15)$$

where $\phi(t) = \phi_b(t) = \sum_{j=0}^{N} b_j t^j, \ b_j \in \mathbb{R}$. (Oberlin-Seeger-B 2009)

- (ii) all “monomial curves”

$$\gamma(t) = (t^{a_1}, t^{a_2}, \ldots, t^{a_d})$$

where $a_1, \ldots, a_d$ are arbitrary real numbers. (Uniform estimates) (Oberlin-Seeger-B 2009)

- (iii) a class of simple curves whose last component satisfies certain monotonicity hypotheses: for example, $\phi(t) = e^{-1/t^a}$, $a > 0$. The torsion may vanish to infinite order here. (Oberlin-Seeger-B 2008)
More general recent results, but with a reduced range of $p$

- Dendrinos-Wright (preprint): general polynomial curves, but only for the Christ’s range of exponents. For example, when $d = 3$, only for the range $p < 5$ (the full range is $p < 7$, when $d = 3$)

- Oberlin (preprint): more general class of curves of simple type; also for the same range of exponents.
Define the adjoint operator $T$ of the restriction operator by

$$Tf(x) = \int_I e^{ix \cdot \gamma_b(t)} f(t) w(t) dt$$

(16)

where $\gamma_b(t)$ is a curve of simple type whose last component is a polynomial $\phi_b(t)$ of degree $\leq N$.

We may show the dual estimate, and it suffices to prove

$$\|Tf\|_{L^q,\infty(B(0,\lambda))} \leq A \|f\|_{L^q(wdt)}$$

(17)

with $A$ independent of $\lambda > 1$. 

The adjoint operator restricted to a $\lambda$-ball, rescaled

\[
T_\lambda f(x) = \psi(x) \int_I e^{i\lambda x \cdot \gamma_b(t)} f(t) w(t) dt
\]

(18)

where $\psi(x)$ is a nonnegative cutoff function and $I = (0, \infty)$. We may assume that $f$ is supported in $(0, 1)$. Put

\[
A_\lambda = \lambda^{d/q} \cdot \sup_b \| T_\lambda \|_{L^q(wdt) \to L^{q,\infty}(\mathbb{R}^d)}
\]

(19)

Namely, $A_\lambda$ is the smallest constant such that

\[
\| T_\lambda f \|_{L^{q,\infty}(\mathbb{R}^d)} \leq \lambda^{-d/q} A_\lambda \| f \|_{L^q(wdt)}.
\]

(20)

Goal: we want to show $A_\lambda \leq C$. It suffices to show

\[
A_\lambda \leq C \cdot A_\lambda^c
\]

for some $c < 1$.  

A $d$-linear operator

- Consider the $d$-fold product of $T_\lambda$, i.e. the $d$-linear expression

$$\prod_{j=1}^{d} T_\lambda f_j(x) = \int_{I^d} e^{ix \cdot [\gamma(t_1) + \cdots + \gamma(t_d)]} \prod_{j=1}^{d} [f_j(t_j) w(t_j)] \, dt_1 \cdots dt_d.$$ 

It suffices to consider the $d$-linear operator $M_\lambda(f_1, \cdots, f_d)(x)$ which is obtained by restricting the domain of integration to $0 < t_1 < \cdots < t_d$.

- Next, decompose $M_\lambda$ using the size of the Vandermonde determinant $v(h) = V(t_1, \cdots, t_d) = \prod_{1 \leq i < j \leq d} |t_i - t_j|$. Let $S_n = \{ h \in I^{d-1} : 2^{-n-1} \leq v(h) < 2^{-n} \}$ and write

$$M_\lambda(f_1, \cdots, f_d)(x) = \sum_{n \in \mathbb{Z}} M_{\lambda, n}(f_1, \cdots, f_d)(x).$$
where

\[ M_{\lambda,n}(f_1, \cdots, f_d)(x) = \]

\[ \psi(x)^d \int_{S_n} \int_I e^{i\lambda x \cdot \Gamma(t,h)} \cdot \prod_{j=1}^{d} [f_j(t + k_j)w(t + k_j)] \ dt \ dh \]

Here \( \Gamma(t, h) = \sum_{j=1}^{d} \gamma(t_j) = \sum_{j=1}^{d} \gamma(t + k_j) \) with

\( h = (h_1, \cdots, h_{d-1}), \ h_j \in I_0, \ k_1 = 0, \ k_2 = h_1, \) and

\( k_j = h_1 + \cdots + h_{j-1}, \ 2 \leq j \leq d. \) (\( \Gamma(t, h) \) is an “offspring curve” of \( \gamma(t) \) in the terminology of Drury-Marshall 1985.)

A fact: \( |S_n| \leq C2^{-2n/d}. \)
The change of variables $y = \Gamma(t, h)$ will make it look like the Fourier transform of a function on $\mathbb{R}^d$. And the Jacobian $J(t, h)$ has the following lower bound on some intervals $I_1, \ldots, I_{C(d, N)}$:

The Jacobian estimate

$$J(t, h) \geq c \nu(h) \prod_{j=1}^{d} \tau(t + k_j)^{1/d}$$

$$= c \nu(h) \prod_{j=1}^{d} w(t + k_j)^{(d+2)/2}$$

for some uniform constant $c = c(d, N) > 0$. 

\[ \text{(21)} \]

\[ \text{(22)} \]
An estimate for the torsion the offspring curve:

- If $\tau(t, h)$ denotes the torsion of the offspring curve $\Gamma(t, h)$, and $W(t, h) = |\tau(t, h)|^{2/(d^2+d)}$, then

\[
W(t, h) \geq c \prod_{j=1}^{d} w(t + k_j)^{1/d}
\]  \hspace{1cm} (23)

- For the cases under consideration we actually get a stronger estimate obtained by replacing the geometric mean on RHS by the arithmetic mean. This implies

\[
W(t, h) \geq c \prod_{j=1}^{d} w(t + k_j)^{a_j}
\]  \hspace{1cm} (24)

for any $a_j \geq 0$ with $a_1 + \cdots + a_d = 1$. (Previously, only the geometric mean was used.)
One of the difficulties which arise in the argument: how should one handle Lorentz norms with weights?

- For instance,

\[ \| f(t)w(t)^{1/p} \|_{L^p(dt)} = \| f \|_{L^p(wdt)}. \] (25)

- On the other hand,

\[ \| f(t)w(t)^{1/p} \|_{L^p,w^1(dt)} \neq \| f \|_{L^p,w^1(dt)}. \] (26)

- To overcome this difficulty, one needs to improve \( L^{p,1} \) to \( L^p \) somehow.

- We can use some multilinear real interpolation methods. In particular, it turns out that a suitable extension of the “multilinear trick” of Christ works.
Introduction of a sequence space in the argument

- For a sequence of functions $f = \{f_m\}$, define
  \[ \|f\|_{\ell_p(X)} = \left( \sum_m [2^{ma} \|f_m\|_X]^p \right)^{1/p}. \]

- Decompose $\mathbb{R}^d$ into $\Omega_m = \{t : 2^m < w(t) \leq 2^{m+1}\}, \ m \in \mathbb{Z}$. So $\sum_{m \in \mathbb{Z}} \chi_{\Omega_m} = 1$.

- Then
  \[ \|f(t)w(t)^a\|_{L_p,1(dt)} \lesssim \sum_m 2^{ma} \|f\chi_{\Omega_m}\|_{L_p,1(dt)} = \|\{f\chi_{\Omega_m}\}\|_{\ell_1^a(L_p,1(dt))} \]  \hspace{1cm} (27)

- We want to improve the latter norm to the following:
  \[ \|f\chi_{\Omega_m}\|_{\ell_{1/p}^p(L_p,p(dt))} = \|f\chi_{\Omega_m}\|_{\ell_{1/p}^p(L_p(dt))} \approx \|f\|_{L_p(wdt)} \]  \hspace{1cm} (28)

- which follows from
  \[ \|f\chi_{\Omega_m}\|_{\ell_{1/p}^p(L_p(dt))}^p = \sum_m 2^m \int |f|^p \chi_{\Omega_m} dt \approx \int |f|^p w(t) dt. \]  \hspace{1cm} (29)
The real interpolation spaces

Let $X$, $Y$ be two (compatible) Banach spaces or more generally quasi-normed spaces. The real interpolation space $(X, Y)_{t, q}$ may be defined by the $K$-method, or equivalently by the $J$-method.

For example,

$$(L^1, L^\infty)_{t, q} = L^{p(q)},$$

where $1/p = (1 - t)/1 + t/\infty$, $0 < t < 1$. 

Two estimates:

- By the Plancherel theorem,

\[
\| M_{\lambda,k}(f_1, \cdots, f_d) \|_2 \leq C \lambda^{-d/2} 2^{k(d-2)/(2d)} \prod_{j=1}^{d} \| f_j \|_{\ell_1^a(L_j^r(dt))}.
\]  

(30)

- From the induction hypothesis (or by the definition of \( A_\lambda \)),

\[
\| M_{\lambda,k}(f_1, \cdots, f_d) \|_{q,\infty} \leq C \lambda^{-d/q} A_\lambda \cdot 2^{-2k/d} \prod_{j=1}^{d} \| f_j \|_{\ell_{\alpha_j}^1(L^q_j(dt))}.
\]

(31)
Lemma

Let $W$ be an $r$-convex space for some $r \in (0, 1)$ or a Banach space (i.e. $r = 1$). (For example, $W = L^{r, \infty}$ for some $r \in (0, 1)$.) Let $X$ and $Y$ be compatible quasi-normed spaces. Suppose that

$$\| Tf \|_W \leq C \| f \|_X^{1-\lambda} \| f \|_Y^\lambda$$

for some $0 < \lambda < 1$. Then

$$\| Tf \|_W \leq C \| f \|_{(X,Y)_{\lambda,r}}.$$  \hspace{1cm} (32)

(When $W$ is a Banach space, we get this conclusion for $r = 1$.)
By summing two geometric series using these two estimates, we get

\[
\| M_\lambda(f_1, \cdots, f_d) \|_{q/d, \infty} \leq C \lambda^{-d^2/q} A_\lambda^{(d-2)/(d+2)} \times \\
\times \prod_{j=1}^{d} \| f_j \|^{(d-2)/(d+2)} X \| f_j \|^{4/(d+2)} Y.
\]

Here \( X := \ell_{\alpha_j}^1(L^{q_j}(dt)) \) and \( Y := \ell_a^1(L^{r_j}(dt)) \). It follows by the previous lemma that

\[
\| M_\lambda(f_1, \cdots, f_d) \|_{q/d, \infty} \leq C \lambda^{-d^2/q} A_\lambda^{(d-2)/(d+2)} \times \\
\times \prod_{j=1}^{d} \| f_j \|^{(X, Y)_{4/(d+2), 1}}.
\]
By the generalized Hölder’s inequality we obtain
\[
\left\| \prod_{j=1}^{dq} T_\lambda f_j \right\|_{1/d, \infty} \leq C \lambda^{-d^2} \cdot A_\lambda^{q(d-2)/(d+2)} \times \\
\times \| f_1 \|_{\ell^r_{\delta_1}(L^{p_1,r})} \| f_2 \|_{\ell^r_{\delta_2}(L^{p_2,r})} \prod_{j=3}^{dq} \| f_j \|_{\ell^r_{\delta_3}(L^{p_2,r})}
\]

We want to use the \( r \)-convexity of the space \( L^{r,\infty} \), \( 0 < r < 1 \).
\[
\left\| \sum_{j=1}^{N} f_j \right\|_{L^{r,\infty}} \leq C \sum_{j=1}^{N} \| f_j \|_{L^{r,\infty}}
\]
where \( C \) is independent of \( N \). (Remark. This fails for \( r = 1 \), i.e. for the weak \( L^1 \)-space.)
Lemma

Let $n \geq 3$ and $W = L^r, \infty$ for some $r \in (0, 1)$. Suppose that $X$ and $Z$ are compatible quasi-normed spaces. Let $\delta_1, \cdots, \delta_n \in \mathbb{R}$, be such that

“not all of $\delta_2, \cdots, \delta_n$ are equal (or $\delta_2 \neq \delta_j$ for $j \geq 3$).”

Suppose also that $S$ is a symmetric $n$-linear operator such that

$$\|S(f_1, \cdots, f_n)\|_W \leq A \|f_1\|_{\ell^r_{\delta_1}(X)} \|f_2\|_{\ell^r_{\delta_2}(Z)} \cdots \|f_n\|_{\ell^r_{\delta_n}(Z)}.$$  \hfill (33)

Then we have

$$\|S(f_1, \cdots, f_n)\|_W \leq C A \prod_{j=1}^n \|f_j\|_{\ell^q_{s}((Z,X)_{\lambda,q})}$$

where $q = nr$, $s = (\delta_1 + \delta_2 + \cdots + \delta_n)/n$, and $\lambda = 1/n$. 
A preliminary lemma

Lemma

Assume the hypotheses of the previous lemma. If $\delta_2 \neq \delta_3$ and

$$\| S(f_1, f_2, f_3) \|_{W} \leq A \| f_1 \|_{r_{\delta_1}}((Z,X)_{\alpha_1,r}) \| f_2 \|_{r_{\delta_2}}((Z,X)_{\alpha_2,r}) \| f_3 \|_{r_{\delta_3}}((Z,X)_{\alpha_3,r}).$$

Then there exists a small number $\varepsilon > 0$ such that

$$\| S(f_1, f_2, f_3) \|_{W} \leq CA \| f_1 \|_{s_1}((Z,X)_{\lambda_1,r}) \| f_2 \|_{s_2}((Z,X)_{\lambda_2,\infty}) \| f_3 \|_{s_3}((Z,X)_{\lambda_3,\infty})$$

where $s_j$ and $\lambda_j$ are as follows:
Here

\[ s_j = \sum_{k=1}^{3} a_{j,k} \delta_k, \quad \lambda_j = \sum_{k=1}^{3} a_{j,k} \alpha_k = a_{j,1} \]

and

\[ \sum_{k=1}^{3} a_{j,k} = 1, \quad \sum_{j=1}^{3} a_{j,k} = 1 \]

for some choice of \( a_{j,k} \in (0, 1), \ 1 \leq j, k \leq 3, \) with \( a_{j,j} \in (1 - \varepsilon, 1). \)

Think of \( \theta_j = (a_{j,1}, a_{j,2}, a_{j,3}) \) as a point on the triangle with vertices at \( e_1 = (1, 0, 0), \ e_2 = (0, 1, 0), \ e_3 = (0, 0, 1) \) in \( \mathbb{R}^3. \) Then these conditions just say that \( \theta_1 + \theta_2 + \theta_3 = (1, 1, 1). \)
Some useful facts from interpolation theory

- Write $A_0 = (Z, X)_{\lambda_0, r}$ and $A_1 = (Z, X)_{\lambda_1, r}$. Then

$$\left( \ell^r_{s_0}(A_0), \ell^r_{s_1}(A_1) \right)_{t, \infty} \supset \ell^r_s((A_0, A_1)_{t, \infty})$$

where $s = (1 - t)s_0 + ts_1$ and $0 < t < 1$.

- If $\lambda_0 \neq \lambda_1$, then

$$(A_0, A_1)_{t, \infty} = ((Z, X)_{\lambda_0, r}, (Z, X)_{\lambda_1, r})_{t, \infty} = (Z, X)_{\lambda, \infty}$$

for $\lambda = (1 - t)\lambda_0 + t\lambda_1$, $0 < t < 1$.

- Assuming $\lambda_0 = \lambda_1$, write $A = (Z, X)_{\lambda_0, \infty} = (Z, X)_{\lambda_1, \infty}$. If $s_0 \neq s_1$, then

$$\left( \ell^r_{s_0}(A), \ell^r_{s_1}(A) \right)_{t, \infty} = \ell^s_{\infty}(A)$$

for $s = (1 - t)s_0 + ts_1$, $0 < t < 1$. 
Figure: $[\theta_2 + \theta_3]/2 = [(1, 1, 1) - \theta_1]/2$; the "cross-interpolation" step
Thank you!